Hybrid Approach : a Tool for Multivariate Cryptography

Luk Bettale[1] Jean-Charles Faugère Ludovic Perret

LIP6 - SALSA
UPMC, CNRS, INRIA Paris-Rocquencourt

Workshop on Tools for Cryptanalysis 2010
Royal Holloway University of London
June 2010

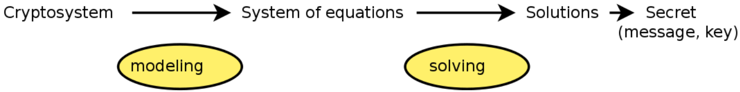
[1] author partially supported by DGA/MRIS (french secretary of defense)

Introduction

Algebraic Cryptanalysis

General analysis (such as linear, differential)

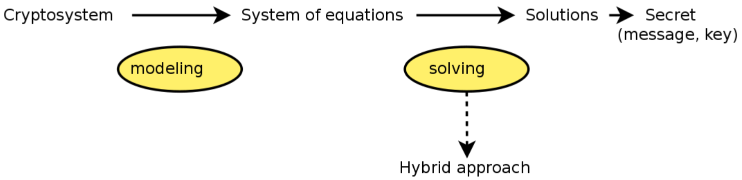
- Modeling
- Solving (or estimate difficulty)



Algebraic Cryptanalysis

General analysis (such as linear, differential)

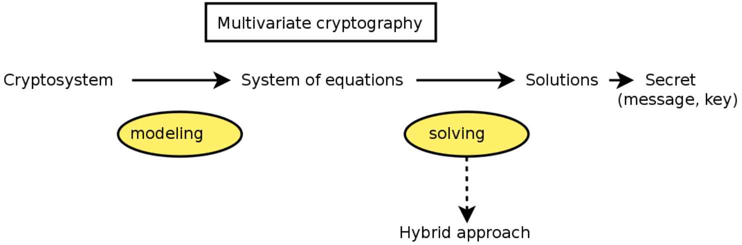
- Modeling
- Solving (or estimate difficulty)



Algebraic Cryptanalysis

General analysis (such as linear, differential)

- Modeling
- Solving (or estimate difficulty)



Setting parameters of multivariate cryptosystems.

Properties

- The public key is a quadratic system
- Very efficient (hardware)
- Resist quantum computers.

Examples

- C^*, HFE
- UOV, SFLASH
- ...

Multivariate signature

Secret key

$$\begin{array}{rcl} \mathbf{F}: \quad \mathbb{F}_q^{n+r} & \rightarrow & \mathbb{F}_q^n \\ (x_1,\ldots,x_{n+r}) & \rightarrow & (f_1(x_1,\ldots,x_{n+r}),\ldots,f_n(x_1,\ldots,x_{n+r})) \end{array}$$

Easy to invert

$$S,T \in \mathrm{GL_{n+r}}(\mathbb{F}_q) \times \mathrm{GL_n}(\mathbb{F}_q)$$

Public key

$$\begin{array}{rcl} \mathbf{G}: \quad \mathbb{F}_q^{n+r} & \rightarrow & \mathbb{F}_q^n \\ (x_1,\ldots,x_{n+r}) & \rightarrow & (g_1(x_1,\ldots,x_n),\ldots,g_n(x_1,\ldots,x_n)) \end{array}$$

$$\mathbf{G} = T \circ \mathbf{F} \circ S = \mathbf{F}(\mathbf{x} \cdot S) \cdot T.$$

$\mathrm{Verify}_{\mathbf{G}}(\mathbf{s},\mathbf{m})$: Evaluate $\mathbf{G}(\mathbf{s}) = \mathbf{m}$

Signature forgery attack

Given a message $\mathbf{m} = (m_1, \ldots, m_n)$, find a signature $(s_1, \ldots, s_{n+r})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{m}$.

Signature forgery attack

Given a message $\mathbf{m} = (m_1, \ldots, m_n)$, find a signature $(s_1, \ldots, s_{n+r})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{m}$.

Solve the system

$$\left\{ \begin{array}{c} g_1(x_1, \ldots, x_{n+r}) - m_1 = 0 \\ \vdots \\ g_n(x_1, \ldots, x_{n+r}) - m_n = 0 \end{array} \right.$$

Signature forgery attack

Given a message $\mathbf{m} = (m_1, \ldots, m_n)$, find a signature $(s_1, \ldots, s_{n+r})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{m}$.

Solve the system

$$\left\{ \begin{array}{c} g_1(x_1, \ldots, x_n, y_1, \ldots, y_r) - m_1 = 0 \\ \vdots \\ g_n(x_1, \ldots, x_n, y_1, \ldots, y_r) - m_n = 0 \end{array} \right.$$

Signature forgery attack

Given a message $\mathbf{m} = (m_1, \ldots, m_n)$, find a signature $(s_1, \ldots, s_{n+r})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{m}$.

Solve the system

$$\begin{cases} g'_1(x_1, \ldots, x_n) - m_1 = 0 \\ \vdots \\ g'_n(x_1, \ldots, x_n) - m_n = 0 \end{cases}$$

An Braeken, Bart Preneel, and Christopher Wolf.
A Study of the Security of Unbalanced Oil and Vinegar Signature Schemes.
CT-RSA 05.

Signature forgery attack

Given a message $\mathbf{m} = (m_1, \ldots, m_n)$, find a signature $(s_1, \ldots, s_{n+r})$ such that $\mathbf{G}(\mathbf{x}) = \mathbf{m}$.

Solve the system

$$\begin{cases} g'_1(x_1, \ldots, x_n) - m_1 = 0 \\ \vdots \\ g'_n(x_1, \ldots, x_n) - m_n = 0 \end{cases}$$

An Braeken, Bart Preneel, and Christopher Wolf.
A Study of the Security of Unbalanced Oil and Vinegar Signature Schemes.
CT-RSA 05.

Lih-Chung Wang, Yuh-Hua Hu, Feipei Lai, Chun-Yen Chou, and Bo-Yin Yang.
Tractable Rational Map Signature.
PKC 05.

TRMS: $q = 2^8, n = 20$.

Given $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ of $\mathbb{F}_q[x_1, \ldots, x_n]$, does there exist $z_1, \ldots, z_n \in \mathbb{F}_q^n$ such that:

$$\begin{cases} f_1(z_1, \ldots, z_n) = 0 \\ \vdots \\ f_m(z_1, \ldots, z_n) = 0 \end{cases}$$

Given $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ of $\mathbb{F}_q[x_1,\ldots,x_n]$, does there exist $z_1,\ldots,z_n \in \mathbb{F}_q^n$ such that:

$$\begin{cases} f_1(z_1,\ldots,z_n) = 0 \\ \vdots \\ f_m(z_1,\ldots,z_n) = 0 \end{cases}$$

- Polynomial System Solving is NP-hard
- Hard in practice for generic polynomials.

Known methods

- Exhaustive search
- Gröbner bases with/without field equations
- ...

- Exhaustive search
- Gröbner bases with/without field equations
- ...

Algorithms

- Buchberger : the historical algorithm
- F_4 : linear algebra on matrices
- F_5 : no useless computations for semi-regular systems

$$\mathrm{GB} : \mathcal{O}\left(\left(m \cdot \binom{n+d_{\mathrm{reg}}-1}{d_{\mathrm{reg}}}\right)^{\omega}\right), \quad \mathrm{FGLM} : \mathcal{O}\left(n \cdot D^{w}\right),$$

with $2 \leqslant \omega \leqslant 3$, D the number of solutions in $\overline{\mathbb{K}}$.

Jean-Charles Faugère.
A new efficient algorithm for computing Gröbner bases (F_4).
Journal of Pure and Applied Algebra 139, June 1999.

Jean-Charles Faugère.
A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5).
ISSAC 2002, July 2002.

Semi-regular systems

Semi-regular systems

- $g \cdot f_i \in \langle f_1, \ldots, f_{i-1} \rangle \Rightarrow g \in \langle f_1, \ldots, f_{i-1} \rangle$ if $\deg(g \cdot f_i) \leqslant d_{reg}$.
  random system $\Rightarrow$ semi-reg.
- The degree of regularity (d_{reg}) can be known a priori
- The more equations we have, the more d_{reg} decrease (e.g. for quadratic systems)
  $m = n \ \rightarrow \ d_{reg} = n+1$ $\qquad$ $m = n+1 \ \rightarrow \ d_{reg} = \lceil \frac{n+1}{2} \rceil$

Magali Bardet, Jean-Charles Faugère, Bruno Salvy and Bo-Yin Yang.
Asymptotic Behaviour of the Degree of Regularity of Semi-Regular Polynomial Systems.
MEGA 2005.

Solving a system

$f_i \in \mathbb{F}_q[x_1, \ldots, x_n]$ for $1 \leqslant i \leqslant n$

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, \ldots, x_n) = 0 \end{cases}$$

Solving a system

$f_i \in \mathbb{F}_q[x_1, \ldots, x_n]$ for $1 \leqslant i \leqslant n$

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, \ldots, x_n) = 0 \end{cases}$$

Specificity $(m = n)$

- Square systems $\Rightarrow d^n$ solutions in the algebraic closure
- $\mathbb{F}_q$ is finite and rather big (no field equations).

Hypotheses

- Regular system $\Rightarrow d_{reg} = n(d-1)+1$
- Semi-regular sub-systems.

Solving a system – Hybrid approach

Solution

We specialize k variables of the system (exhaustive search)
$\Rightarrow$ the system becomes over-defined

+ The degree of regularity decreases
+ The number of solutions is 0 or 1
– We have to compute q^k Gröbner bases.

Luk Bettale, Jean-Charles Faugère and Ludovic Perret.
Hybrid approach for solving multivariate systems over finite fields.
In Journal of Mathematical Cryptology, Volume 3, issue 3. Sep 2009.

Solution

We specialize k variables of the system (exhaustive search)
$\Rightarrow$ the system becomes over-defined

+ The degree of regularity decreases
+ The number of solutions is 0 or 1
– We have to compute q^k Gröbner bases.

Luk Bettale, Jean-Charles Faugère and Ludovic Perret.
Hybrid approach for solving multivariate systems over finite fields.
In Journal of Mathematical Cryptology, Volume 3, issue 3. Sep 2009.

A tradeoff between exhaustive search and Gröbner bases computation.

Complexity analysis

Proposition

Let $\mathbb{F}_q$ *be a finite field and* $\{f_1, \ldots, f_n\} \subset \mathbb{F}_q[x_1, \ldots, x_n]$ *be a semi-regular system of equations of degree* d*.*

$$\mathcal{O}\left(\underbrace{\min_{0 \leqslant k \leqslant n}}_{\textit{tradeoff}}\left(\underbrace{q^k}_{\textit{exh. search}}\underbrace{\left(n \cdot \binom{n-k-1+\mathrm{d_{reg}}(n-k,n,d)}{\mathrm{d_{reg}}(n-k,n,d)}\right)^{\omega}}_{GB}\underbrace{+n \cdot D^{\omega}}_{FGLM}\right)\right),$$

where $2 \leqslant \omega \leqslant 3$*.*

$\mathrm{d_{reg}}(n, m, d)$ is the d_{reg} of a semi-regular system of m equations of degree d in n variables.

Proposition

Let $\mathbb{F}_q$ be a finite field and $\{f_1, \ldots, f_n\} \subset \mathbb{F}_q[x_1, \ldots, x_n]$ be a semi-regular system of equations of degree d.

$$\mathcal{O}\left(\underbrace{\min_{0 \leqslant k \leqslant n}}_{\textit{tradeoff}}\left(\underbrace{q^k}_{\textit{exh. search}}\underbrace{\left(n \cdot \binom{n-k-1+\mathrm{d_{reg}}(n-k,n,d)}{\mathrm{d_{reg}}(n-k,n,d)}\right)^{\omega}}_{GB}\underbrace{+n \cdot D^{\omega}}_{FGLM}\right)\right),$$

where $2 \leqslant \omega \leqslant 3$.

$\mathrm{d_{reg}}(n, m, d)$ is the d_{reg} of a semi-regular system of m equations of degree d in n variables.

The degree of regularity can be computed exactly.

Proposition

Let $\mathbb{F}_q$ *be a finite field and* $\{f_1,\ldots,f_n\} \subset \mathbb{F}_q[x_1,\ldots,x_n]$ *be a semi-regular system of equations of degree* d*.*

$$\mathcal{O}\left(\underbrace{\min_{1\leqslant k\leqslant n}}_{\textit{tradeoff}}\left(\underbrace{q^k}_{\textit{exh. search}}\ \underbrace{\left(n\cdot\binom{n-k-1+\mathrm{d_{reg}}(n-k,n,d)}{\mathrm{d_{reg}}(n-k,n,d)}\right)^{\omega}}_{GB}\right)\right),$$

where $2 \leqslant \omega \leqslant 3$*.*

$\mathrm{d_{reg}}(n,m,d)$ is the d_{reg} of a semi-regular system of m equations of degree d in n variables.

The degree of regularity can be computed exactly.

Asymptotic analysis $(d = 2)$

Approximation of $\mathrm{d}_{\mathrm{reg}}(n-k, n, 2)$

$$d_{reg} \sim \frac{n+k}{2} - \sqrt{nk} + \mathcal{O}((n-k)^{1/3})$$

when $n \to \infty$.

Magali Bardet

Étude des systèmes algébriques surdéterminés. Applications aux codes correcteurs et à la cryptographie.

Ph.D. thesis, Université de Paris VI, 2004.

Approximation of the complexity

$$C_{Hyb} = \mathcal{O}\left(q^k \left(\frac{n}{\sqrt{2\pi}}\right)^{\omega} \left(\frac{\left(\frac{3n-k}{2} - 1 - \sqrt{nk}\right)^{(3n-k-1)/2-\sqrt{nk}}}{(n-k-1)^{(n-k-1/2)}\left(\frac{n+k}{2} - \sqrt{nk}\right)^{(n+k+1)/2-\sqrt{nk}}} \right)^{\omega} \right)$$

when $n \to \infty$.

Borderline case ($d = 2$)

Classical approach

($d_{reg} = n+1$)

$$\mathcal{O}\left(\left(n \cdot \binom{2n}{n-1}\right)^{\omega}\right)$$

Hybrid approach with $k = 1$

($d_{reg} = \lceil \frac{n+1}{2} \rceil$)

$$\mathcal{O}\left(q\left(n \cdot \binom{3(n-1)/2}{n-2}\right)^{\omega}\right)$$

Best tradeoff > 0

$$\log_2(q) \leq 0.6226 \cdot \omega \cdot n + \mathcal{O}(\log_2(n))$$

when $n \to \infty$.

Borderline case $(d = 2)$

Classical approach

$(d_{reg} = n+1)$

$$\mathcal{O}\left(\left(n \cdot \binom{2n}{n-1}\right)^{\omega}\right)$$

Hybrid approach with $k = 1$

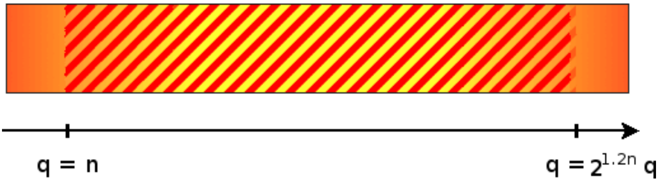
$(d_{reg} = \lceil \frac{n+1}{2} \rceil)$

$$\mathcal{O}\left(q\left(n \cdot \binom{3(n-1)/2}{n-2}\right)^{\omega}\right)$$

Best tradeoff > 0

$$\log_2(q) \leq 0.6226 \cdot \omega \cdot n + \mathcal{O}(\log_2(n))$$

when $n \to \infty$.



Find the best tradeoff by solving $\dfrac{\partial \log(C_{Hyb})}{\partial k} = 0$.

$$\log(q) + \omega\left(\log(n-k-1) + \frac{1}{2(n-k-1)}\right)$$

$$-\frac{\omega}{2}(1+\sqrt{n/k})\left(\log\left(\frac{3n-k}{2} - 1 - \sqrt{nk}\right) + \frac{1}{2\left(\frac{3n-k}{2} - 1 - \sqrt{nk}\right)}\right)$$

$$-\frac{\omega}{2}(1-\sqrt{n/k})\left(\log\left(\frac{n+k}{2} - \sqrt{nk}\right) + \frac{1}{2\left(\frac{n+k}{2} - \sqrt{nk}\right)}\right) = 0.$$

Finding the best tradeoff $(d = 2)$

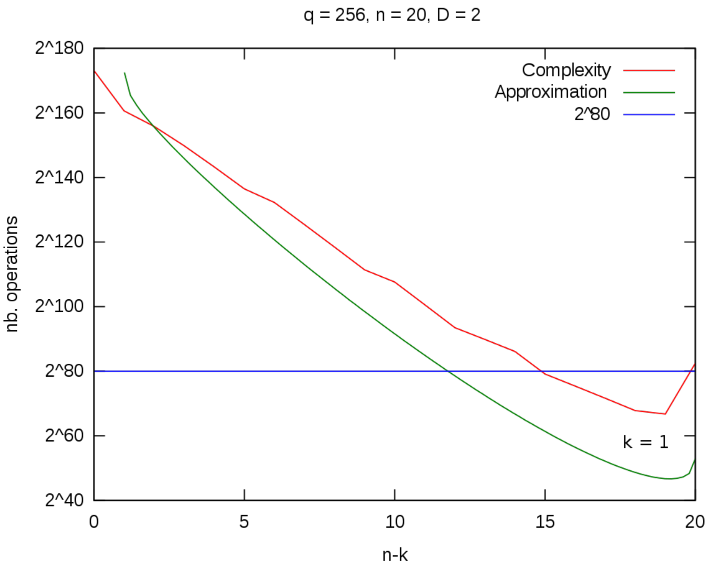
Find the best tradeoff by solving $\frac{\partial \log(C_{Hyb})}{\partial k} = 0.$

$$k \approx \frac{n}{c^2}$$

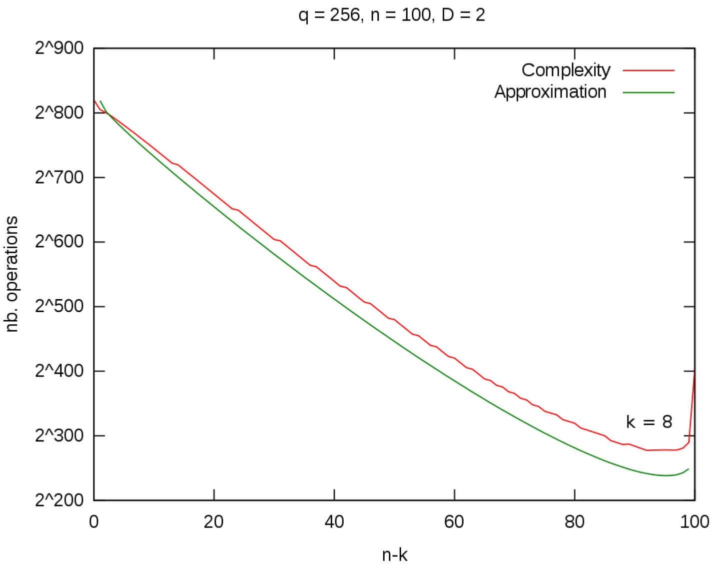
$$8\,q\,(c-1)^{3\,c-3}\,e^{-3/2\,c\ln((3\,c+1)(c-1))}\,(c-1)^3\,(c+1)^3 \\ -((3\,c+1)\,(c-1))^{3/2} = 0$$

q	2	16	256	65521	2^{32}	2^{64}	2^{80}
c^2	1.23	3.07	9.15	37.13	160.37	678.32	1073.1

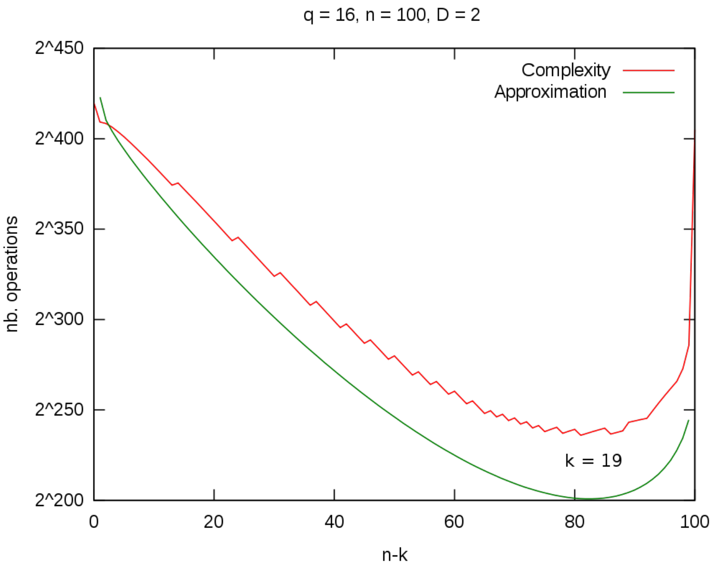
Comparison


q = 256, n = 20, D = 2
Complexity
Approximation
2^80
k = 1
nb. operations
2^180
2^160
2^140
2^120
2^100
2^80
2^60
2^40
0
5
10
15
20
n-k

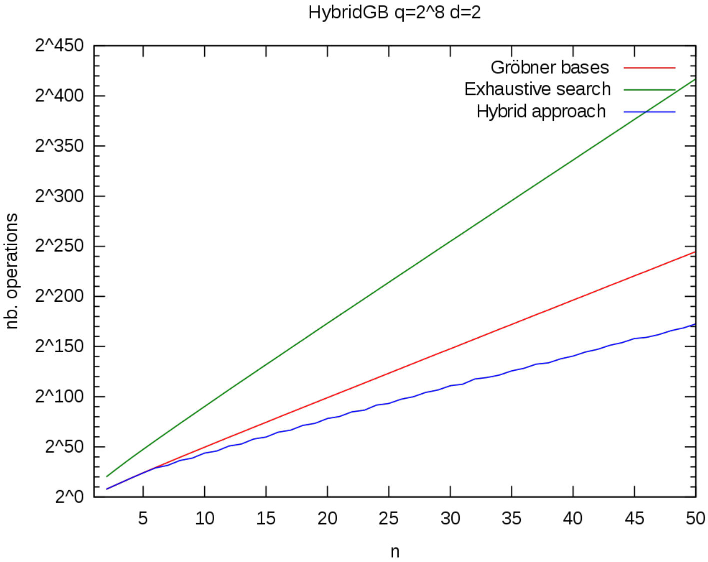
Comparison


q = 256, n = 100, D = 2
Complexity
Approximation
k = 8
nb. operations
2^900
2^800
2^700
2^600
2^500
2^400
2^300
2^200
0
20
40
60
80
100
n-k

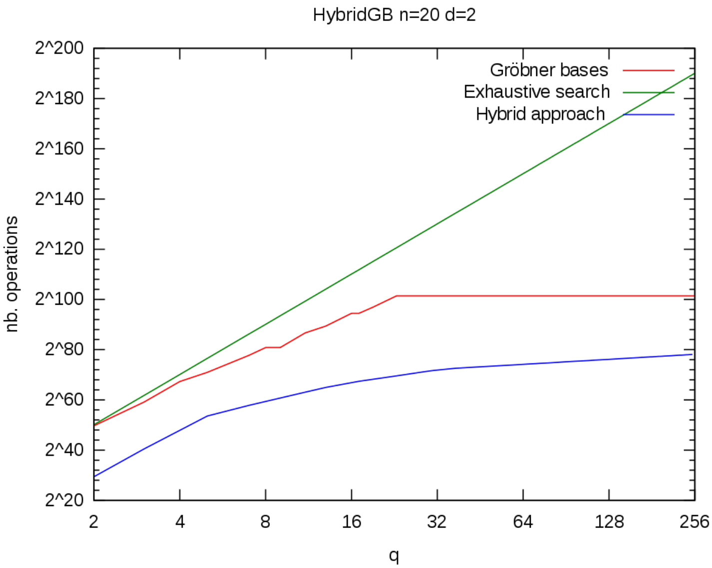
Comparison


q = 16, n = 100, D = 2
Complexity
Approximation
k = 19
nb. operations
2^450
2^400
2^350
2^300
2^250
2^200
0
20
40
60
80
100
n-k

Overall


HybridGB q=2^8 d=2
nb. operations
2^450
2^400
2^350
2^300
2^250
2^200
2^150
2^100
2^50
2^0
5
10
15
20
25
30
35
40
45
50
n
Gröbner bases
Exhaustive search
Hybrid approach

Overall


HybridGB n=20 d=2
Gröbner bases
Exhaustive search
Hybrid approach
nb. operations
2^200
2^180
2^160
2^140
2^120
2^100
2^80
2^60
2^40
2^20
2
4
8
16
32
64
128
256
q

Input: $\mathbb{K}$ is finite, $\{f_1, \ldots, f_m\} \subset \mathbb{K}[x_1, \ldots, x_n]$ is zero-dimensional, $k \in \mathbb{N}$.

Output: $\mathcal{S} = \{(z_1, \ldots, z_n) \in \mathbb{K}^n : f_i(z_1, \ldots, z_n) = 0, 1 \leq i \leq m\}$.

$\mathcal{S} := \emptyset$

for all $(v_1, \ldots, v_k) \in \mathbb{K}^k$ **do**

Find the set of solutions $\mathcal{S}' \subset \mathbb{K}^{(n-k)}$ of

$$\begin{cases} f_1(x_1, \ldots, x_{n-k}, v_1, \ldots, v_k) = 0 \\ \vdots \\ f_m(x_1, \ldots, x_{n-k}, v_1, \ldots, v_k) = 0 \end{cases}$$

using the zero-dim solving strategy.

$\mathcal{S} := \mathcal{S} \cup \{(z'_1, \ldots, z'_{n-k}, v_1, \ldots, v_k) : (z'_1, \ldots, z'_{n-k}) \in \mathcal{S}'\}$.

end for

return $\mathcal{S}$.

```
function HybridSolving(F,k)
    R := Universe(F); K := BaseRing(R); n := Rank(R);
    Rp<[x]> := PolynomialRing(K,n-k);
    Kev := VectorSpace(K,k);
    S := [ ];
    for e in Kev do
        v := Eltseq(e);
        fp := [ Evaluate(f,x cat v) : f in F ];
        Sp := VarietySequence(Ideal(fp));
        S cat:= [ s cat v : s in Sp ];
    end for;
    return S;
end function;
```

http://www-salsa.lip6.fr/~bettale/hybrid.html

TRMS: Experimental results

q	n	k	T_{F_5}	mem. (MB)	Nop_{F_5}	Nop
2^8	20	1	-	-	-	-
		2	51h	41940	2^{41}	2^{57}
		3	2h45	4402	2^{37}	2^{61}
		4	626s	912	2^{34}	2^{66}

Practical tradeoff : $k = 2$. Broken in $<$ 51h on 2^{16} proc.

Luk Bettale, Jean-Charles Faugère, and Ludovic Perret.
Cryptanalysis of the TRMS Signature Scheme of PKC'05.
AFRICACRYPT 2008.

	n	q	expected security	Gröbner basis $(k=0)$	hybrid approach	mem.
UOV_{30}	10	2^8	2^{80}	2^{41}	2^{37} $(k=1)$	2 MB
UOV_{60} / enTTS	20	2^8	2^{160}	2^{82}	2^{66} $(k=1)$	139 GB
					2^{67} $(k=2)$	12 GB
Rainbow / amTTS	24	2^8	2^{192}	2^{98}	2^{78} $(k=1)$	10 TB
					2^{79} $(k=2)$	816 GB

Andrey Bogdanov, Thomas Eisenbarth, Andy Rupp, and Christopher Wolf.
Time-Area Optimized Public-Key Engines: MQ-Cryptosystems as Replacement for Elliptic Curves?
CHES '08: Proceedings of the 10th international workshop on Cryptographic Hardware and Embedded Systems.

Analysis of several multivariate schemes

	n	q	expected security	Gröbner basis ($k=0$)	hybrid approach	mem.
UOV_{30}	10	2^8	2^{80}	2^{41}	2^{37} ($k=1$)	2 MB
UOV_{60}	20	2^8	2^{160}	2^{82}	2^{66} ($k=1$)	139 GB
enTTS					2^{67} ($k=2$)	12 GB
Rainbow	24	2^8	2^{192}	2^{98}	2^{78} ($k=1$)	10 TB
amTTS					2^{79} ($k=2$)	816 GB

Andrey Bogdanov, Thomas Eisenbarth, Andy Rupp, and Christopher Wolf.
Time-Area Optimized Public-Key Engines: MQ-Cryptosystems as Replacement for Elliptic Curves?
CHES '08: Proceedings of the 10th international workshop on Cryptographic Hardware and Embedded Systems.

Block hybrid approach: Motivations

High degree polynomials

Semaev polynomials

Solving DLP on curves

Pierrick Gaudry.
Index calculus for abelian varieties of small dimension and the elliptic curve discrete logarithm problem.
Journal of Symbolic Computation 2009.

Previous approaches

Field equations

$$\langle f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n),x_1^q-x_1,\dots,x_n^q-x_n\rangle$$

$$x^q-x=\prod_{i=1}^{d}(x-e_{1,i})\dots\prod_{i=1}^{d}(x-e_{l,i})$$

Hybrid approach

$$\mathcal{I}=\langle f_1(x_1,\dots,x_{n-k},v_1,\dots,v_k),\dots,f_m(x_1,\dots,x_{n-k},v_1,\dots,v_k)\rangle$$

$$\mathcal{J}=\langle f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n),x_{n-k+1}-v_1,\dots,x_n-v_k\rangle$$

$$\mathcal{I}=\mathcal{J}\cap\mathbb{K}[x_1,\dots,x_{n-k}]$$

Block hybrid approach

Principle

$$\langle f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n),\prod_{i=1}^{d}(x_1-e_{1,i}),\dots,\prod_{i=1}^{d}(x_k-e_{k,i})\rangle$$

We have to compute $\left(\frac{q}{d}\right)^k$ Gröbner bases.

Complexity

$$\mathcal{O}\left(\min_{0\leqslant k\leqslant n}\left\lceil\frac{q}{d}\right\rceil^k\cdot \mathrm{C}_{\mathrm{F}_5}\left(n,\{d_1,\dots,d_m,\underbrace{d,\dots,d}_{k}\}\right)\right)$$

Conclusion

Applications in cryptography

- A general tool for solving random systems over finite field
- Reevaluate parameters of multivariate cryptosystems
- Block hybrid approach for high degree equations
- Implementation in MAGMA.
  `http://www-salsa.lip6.fr/~bettale/hybrid.html`